Three Dimensional Geometry

Multiple Choice Questions

Q: 1 **Given below are the vector equations of two planes.**

$$\left[\vec{r} - \left(4\hat{i} + 3\hat{j} - 4\hat{k}\right)\right] \cdot \left(\hat{i} + 4\hat{j} - \hat{k}\right) = 0$$

$$\left[\vec{r} - \left(6\hat{i} + \hat{j} - 5\hat{k}\right)\right] \cdot \left(\hat{i} + 4\hat{j} - \hat{k}\right) = 0$$

Which of the following is true about these planes?

1. They coincide with each other.
2. They are parallel to each other.
3. They are perpendicular to each other.
4. They are not perpendicular but intersect each other along a unique line.

Q: 2 **Shown below are equations of four planes.**

P_1 **: -2 x - 3 y - 4 z - 20 = 0**
P_2 **: $2x + 3y + 4z$ - 9 = 0**
P_3 **: -4 x - 6 y + 8 z = 0**
P_4 **: 4 x + 6 y + 8 z = 0**

Which of the provided planes does NOT share parallelism with the rest?

1. P_1
2. P_2
3. P_3
4. P_4

Free Response Questions

Q: 3 **There are three points P, Q and R. The direction ratios of line PQ are (-2), 1 and 3 and that of line QR are 6, (-3) and (-9).** **[1]**

Is there a unique plane that contains the points P, Q and R? Justify your answer.

Q: 4 **The equations for line *m* and line *n* are given below.** **[1]**

Line *m*: $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{2}$

Line *n*: $\frac{x-1}{2} = \frac{y-1}{1} = \frac{1-z}{k}$

If line *m* is perpendicular to line *n*, find the value of *k*. Show your steps.

(Note: k ≠ 0.)

Q: 5 **Line *m* and line *n* are two lines which are not parallel, and do not intersect each other.** **[1]**

Asha, Ravi and Saquib tried to find the shortest distance between lines *m* and *n*.

♦ Asha found a line *k* parallel to line *m* on the same plane as line *n*. She found the shortest distance as the distance between lines *k* and *n*.

♦ Ravi found the length of a line segment that is perpendicular to both line *m* and line *n* as the shortest distance.

♦ Saquib said that it is impossible to find the distance between the two lines, since the lines are not parallel, and do not intersect.

Which of them is/are using the correct approach? Justify your answer.

Q: 6 **[1]**

$\vec{b}$ is a vector parallel to a line $\vec{c}$.

$\vec{n}$ is the normal from the origin to the plane in which $\vec{c}$ lies.

Find $\vec{b} \cdot \vec{n}$. Give a valid reason.

Q: 7 **Shown below are the equations of two planes.** **[1]**

$P_1 : 3x - 3y - 3z = 1$
$P_2 : -3x + ny - z = 4$

If P_1 is perpendicular to P_2, find the value of n. Show your work.

Q: 8 **A line given by $\frac{x+1}{3} = \frac{y+1}{2} = \frac{2-z}{2}$ is parallel to the plane $6x + 4y + Gz = 12$.** **[2]**

Find the value of G. Show your steps.

Q: 9 **The equation of a plane can be given by $px - 2y + z = q$, where p and q $\in$ R, such that:** **[2]**

♦ the plane is parallel to $-6x + 4y - 2z = 12$.

♦ the point (2, -3, 5) lies on the plane.

Find p and q. Show your work.

Q: 10 Shown below is the equation of a plane P. **[2]**

$\vec{r}.(0\hat{i} - 5\hat{j} + 0\hat{k}) = 8$

WITHOUT using the formula for distance of a point from a plane, find out how far the plane P is from the origin.

Q: 11 **Two lines are given below in vector form:** **[3]**

$\vec{r_1} = (\hat{i} - \hat{j}) + \lambda(-\hat{i} + \hat{j} - \hat{k})$

$\vec{r_2} = (4\hat{i} - 3\hat{j} + \hat{k}) + \mu(-\hat{i} + 2\hat{j} - 3\hat{k})$

Prove that the lines are non-skew lines. Give valid reasons.

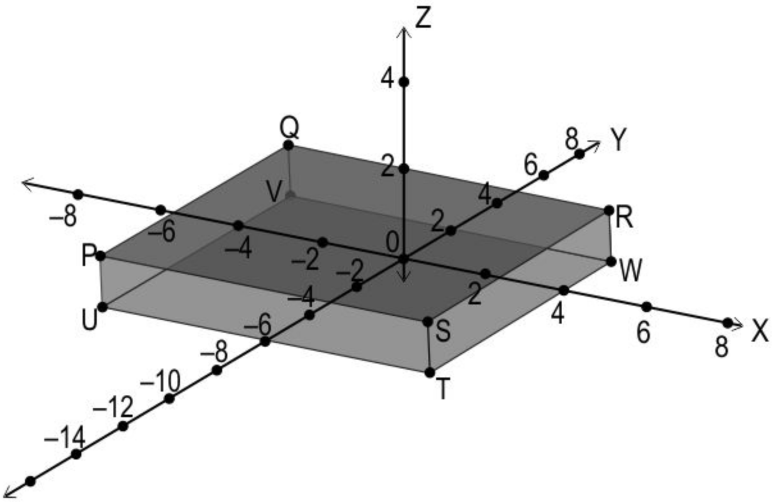
Q: 12 **Shown below are the equations of a line and a plane.** **[3]**

Line: $\vec{r} = \hat{i} + 2\hat{j} + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$
Plane: $8x - 2y + 6z = 1$

where λ is some real number.

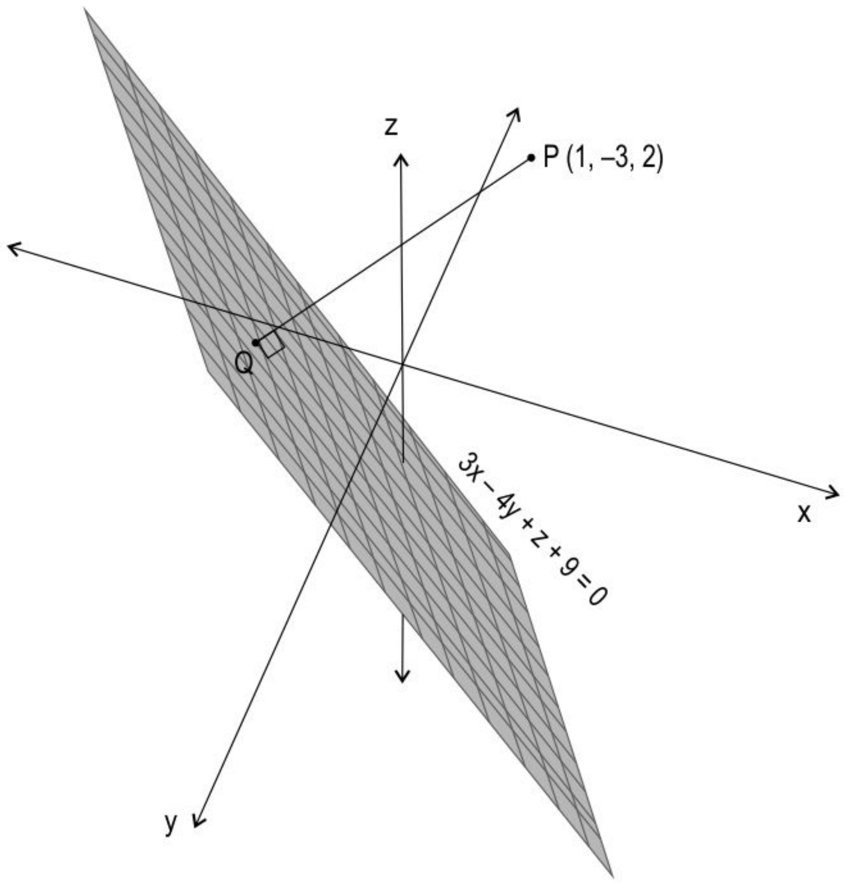
Check whether they intersect. If yes, find the point of intersection. If not, state a valid reason. Show your work.

Q: 13 **A cuboid is shown below, such that face TUVW lies on the *x* - *y* plane. The height of the cuboid is 2 units.** [3]



Find the equation of the line joining P and W. Does it pass through the origin? Show your work and give valid reason(s).

Q: 14 **Point P meets the plane shown below at point Q.** **[5]**



Find the coordinates of Q and the distance PQ. Show your work.

Q: 15 **The equation of line *m* is given by $\frac{x+8}{2} = \frac{5-y}{3} = \frac{z-4}{5}$. A line segment PQ is to be drawn perpendicular to line *m*, such that line *m* bisects PQ.** **[5]**

If P's coordinates are given by (-2, -7, 2), then find the coordinates of point Q. Show your steps.

Q: 16 **Shown below is an equation of a plane, where P is a constant.** **[5]**

$$3y + 4z + P = 0$$

The distance from the origin to the plane is the same as the distance from the point (n, 2, -4) to the plane. The distance from the point (2, m, 4) to the plane is six times the distance from the origin to the plane.

Find all possible values of:
i) P
ii) m
iii) n

Show your work. Give a valid reason.

The table below gives the correct answer for each multiple-choice question in this test.

Q.No	Correct Answers
1	2
2	3

Q.No	What to look for	Marks
3	Finds that the direction ratios of PQ and QR are proportional. The working may look as follows: $\frac{6}{-2} = \frac{-3}{1} = \frac{-9}{3}$ = (-3), where (-3) is the constant of proportionality.	0.5
	States that there is no unique plane because P, Q and R are collinear and infinitely many planes that contain three collinear points exist.	0.5
4	Since the two lines are perpendicular, writes the equation: $3(2) + 2(1) + 2(-k) = 0$	0.5
	Solves the above equation to find the value of k as 4.	0.5
5	Writes that only Ravi is using the correct approach. Gives a reason. For example, lines m and n are skew lines, and the distance between skew lines is given by the length of a line segment that is perpendicular to both lines. Hence, Saquib is incorrect. Asha is incorrect as the shortest distance between two non-parallel lines on the same plane is 0.	1
6	States that the normal to a plane is perpendicular to every line on the plane and the lines parallel to the lines on the plane.	0.5
	Finds the value of the dot product as: $\vec{b} \cdot \vec{n} = 0$	0.5
7	Writes that since P_1 is perpendicular to P_2, their normals are also perpendicular. Thus equates the dot product of the normals to zero and finds the value of n as (-2). The working may look as follows. $3(-3) + (-3)n + (-3)(-1) = 0$ $\Rightarrow -9 - 3n + 3 = 0$ $\Rightarrow n = -2$	1

Q.No	What to look for	Marks
8	**Writes the equation of the line in vector form as follows:** $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{a} = (-\hat{i} - \hat{j} + 2\hat{k})$ and $\vec{b} = (3\hat{i} + 2\hat{j} - 2\hat{k})$	**0.5**
	Finds the equation of the plane in vector form as $\vec{r}.\hat{n} = d$, where $d = 12$, $\hat{n} = \frac{\vec{n}}{\lvert\vec{n}\rvert}$, and $\vec{n} = (6\hat{i} + 4\hat{j} + G\hat{k})$. Hence finds the equation of the plane as $\frac{\vec{r}.(6\hat{i}+4\hat{j}+G\hat{k})}{\lvert\sqrt{6^2+4^2+G^2}\rvert} = 12$	**0.5**
	As the line is parallel to the plane, notes that $\sin\phi = 0$, where ϕ is the angle between the given line and plane. $\therefore\ \vec{b}.\hat{n} = 0.$ Hence finds $\vec{b}.\hat{n} = (3\hat{i} + 2\hat{j} - 2\hat{k}).\frac{(6\hat{i}+4\hat{j}+G\hat{k})}{\lvert\sqrt{6^2+4^2+G^2}\rvert} = 0$ $\Rightarrow (3\hat{i} + 2\hat{j} - 2\hat{k}).(6\hat{i} + 4\hat{j} + G\hat{k}) = 0$ $\Rightarrow 18 + 8 - 2G = 0$	**0.5**
	Solves the above equation to find G = 13.	**0.5**
9	**Finds p = 3 by writing the following as the planes are parallel:** $\frac{p}{-6} = \frac{-2}{4} = \frac{1}{-2}$	**1**
	Finds q = 17 by substituting the coordinates of the point (2, -3, 5) in the equation of the plane as: **3(2) - 2(-3) + (5) = q**	**1**

Q.No	What to look for	Marks
10	**Finds the magnitude of the normal vector to the plane from the origin as:** $\left\|\vec{n}\right\| = \sqrt{0^2 + 5^2 + 0^2} = 5$	0.5
	Rewrites the given equation as: $\vec{r}\,.\frac{(0\hat{i} - 5\hat{j} + 0\hat{k})}{\|\vec{n}\|} = \frac{8}{\|\vec{n}\|}$ $\Rightarrow \vec{r}\,.\,\hat{n} = \frac{8}{5}$	1
	Compares the above with the equation of a plane in normal form and finds the distance between the plane and the origin as $\frac{8}{5}$ units.	0.5
11	**Considers the lines in vector form as:** $\vec{r_1} = \vec{a_1} + \lambda\vec{b_1}$ $\vec{r_2} = \vec{a_2} + \lambda\vec{b_2}$ which denotes that $\vec{r_1}$ passes through the point A, with position vector $\vec{a_1}$, and is parallel to $\vec{b_1}$, and $\vec{r_2}$ passes through the point B, with position vector $\vec{a_2}$, and is parallel to $\vec{b_2}$.	0.5
	Writes the vector of the lines joining A and B as: $\vec{AB} = (1 - 4)\hat{i} + (-1 - (-3))\hat{j} + (0 - 1)\hat{k} = -3\hat{i} + 2\hat{j} - 1\hat{k}$	0.5
	Writes that if the lines are coplanar, they will definitely be non-skew lines.	0.5

Q.No	What to look for	Marks
	Hence, proves coplanarity as follows: $\overrightarrow{AB}.(\overrightarrow{b_1} \times \overrightarrow{b_2}) = 0$ the LHS of which translates to: $\begin{vmatrix} -3 & 2 & -1 \\ -1 & 1 & -1 \\ -1 & 2 & -3 \end{vmatrix}$	0.5
	Finds the value of the determinant as 0.	0.5
	Hence, concludes that the lines are coplanar and therefore non-skew lines.	0.5
12	Rearranges the equation of the line as: $\vec{r} = (1 + \lambda)\hat{i} + (2 - 3\lambda)\hat{j} + 2\lambda\,\hat{k}$ and finds a point P ((1 + λ), (2 - 3λ), 2λ) on the line.	0.5
	Substitutes the coordinates of P in the equation of the plane as: 8(1 + λ) - 2(2 - 3λ) + 6(2λ) = 1	0.5
	Finds the value of λ as ($-\frac{3}{26}$) by simplifying the above equation as follows: 8 +8λ - 4 + 6λ + 12λ = 1 => 4 + 26λ = 1 => $\lambda = -\frac{3}{26}$ Writes that there exists a value of λ for which the point P satisfies the equations of both the line and the plane. Hence concludes that the line and the plane intersect.	1

Q.No	What to look for	Marks
	Substitutes the value of λ in point P to find the point of intersection as: P = ($\frac{23}{26}$, $\frac{61}{26}$, - $\frac{3}{13}$)	1
13	Finds the coordinates of P as (-4, -6, 2). Finds the coordinates of W as (4, 2, 0).	1
	Finds the direction ratios of PW as follows: l = 4 - (-4) = 8 m = 2 - (-6) = 8 n = 0 - 2 = (-2)	0.5
	Assumes that (x, y, z) is an arbitrary point on the line joining P and W. Finds the equation of the line as: $\frac{x+4}{8} = \frac{y+6}{8} = \frac{2-z}{2}$ or $\frac{x-4}{8} = \frac{y-2}{8} = \frac{-z}{2}$	1
	Writes that the line does not pass through the origin, and gives a valid reason. For example, in the equation from step 3, $\frac{0+4}{8} \neq \frac{0+6}{8} \neq \frac{2-0}{2}$	0.5
14	Identifies the direction ratios of the perpendicular to the plane as 3, -4 and 1.	0.5
	Takes Q(x_1, y_1, z_1), identifies that the direction ratios of the perpendicular and the direction ratios of PQ are proportional and writes the following: $\frac{x_1-1}{3} = \frac{y_1+3}{-4} = \frac{z_1-2}{1} = k$ $\Rightarrow x_1 = 3k + 1;\ y_1 = -4k - 3;\ z_1 = k + 2$	1

Q.No	What to look for	Marks
	Substitutes the above values in the equation of the plane to obtain k = -1. The working may look as follows: **$3(3k + 1) - 4(-4k - 3) + 1(k + 2) + 9 = 0$** **$\therefore k = -1$**	**1**
	Hence, finds the coordinates of Q by substitution as Q(-2, 1, 1).	**1**
	Uses the distance formula to express PQ as follows: **PQ = $\sqrt{(1+2)^2 + (-3-1)^2 + (2-1)^2} = \sqrt{3^2 + (-4)^2 + 1^2}$ units**	**1**
	Solves the above to find the distance PQ as $\sqrt{26}$ units.	**0.5**
15	**Assumes that O(x, y, z) is the point of intersection of PQ and line m.** **Writes that $\frac{x+8}{2} = \frac{5-y}{3} = \frac{z-4}{5} = \lambda$, where λ is a constant.** **Using the equations from step 1, finds x, y, and z as follows:** **$x = 2\lambda - 8$** **$y = 5 - 3\lambda$** **$z = 5\lambda + 4$**	**1**
	Notes that O(x, y, z) and P both lie on line PQ. **Finds the direction ratios of PQ as:** **$(2\lambda - 8 - (-2), 5 - 3\lambda - (-7), 5\lambda + 4 - 2)$** **$= (2\lambda - 6, 12 - 3\lambda, 5\lambda + 2)$**	**0.5**
	Since PQ is perpendicular to line m, equates the dot product of their direction ratio to 0 to find λ as 1. **$(2\lambda - 6, 12 - 3\lambda, 5\lambda + 2).(2, -3, 5) = 0$** **$\Rightarrow 2(2\lambda - 6) - 3(12 - 3\lambda) + 5(5\lambda + 2) = 0$** **$\Rightarrow 4\lambda - 12 - 36 + 9\lambda + 25\lambda + 10 = 0$** **$\Rightarrow 38\lambda = 38$** **$\Rightarrow \lambda = 1$**	**1.5**

<table>
<tr><th>Q.No</th><th>What to look for</th><th>Marks</th></tr>
<tr><td></td><td>Finds the O(x,y,z), the coordinates of the foot of the perpendicular from P to line m as (-6, 2, 9).</td><td>0.5</td></tr>
<tr><td></td><td>Assumes the coordinates of Q as (a, b, c). Finds Q as (-10, 11, 16) using the midpoint theorem as:

-2 + a = 2(-6)
=> a = -10

-7 + b = 2(2)
=> b = 11

2 + c = 2(9)
=> c = 16</td><td>1.5</td></tr>
<tr><td>16</td><td>Finds the distance from the origin to the plane as follows:

$D_1 = \frac{|P|}{5}$</td><td>0.5</td></tr>
<tr><td></td><td>Finds the distance from the point (n, 2, -4) to the plane as follows:

$D_2 = \frac{|0(n)+3(2)+4(-4)+P|}{5} = \frac{|-10+P|}{5}$</td><td>0.5</td></tr>
<tr><td></td><td>i) Uses the given condition that $D_1 = D_2$ to find the value of P as 5. The working may look as follows:

| P | = |-10 + P |

<table>
<tr><td>P = -10 + P</td><td>OR</td><td>- P = 10 - P</td><td>OR</td><td>P = 10 - P</td><td>OR</td><td>- P = -10 + P</td></tr>
<tr><td>0 = 10</td><td></td><td>0 = 10</td><td></td><td>=> 2 P = 10</td><td></td><td>=> -2 P = -10</td></tr>
<tr><td>which is not possible.</td><td></td><td>which is not possible.</td><td></td><td>=> P = 5</td><td></td><td>=> P = 5</td></tr>
</table>

(Award full marks if two of the unique cases are shown rather than 4 cases.)</td><td>1.5</td></tr>
<tr><td></td><td>ii) Finds the distance of the point (2, m, 4) from the plane as follows:

$D_3 = \frac{|0(2)+3(m)+4(4)+P|}{5}$</td><td>0.5</td></tr>
</table>

Q.No	What to look for	Marks
	Uses the given condition that $D_3 = 6D_1$ to find the value of *m* as 3 or -17. The working may look as follows: $\frac{\lvert 0(2)+3(m)+4(4)+P \rvert}{5} = \frac{6\lvert P \rvert}{5}$ **=> $\lvert 3m + 16 + 5 \rvert = 30$** $3m + 16 + 5 = 30$ OR $3m + 16 + 5 = -30$ => $3m = 9$ / => $3m = -51$ => $m = 3$ / => $m = -17$	1
	iii) Writes that *n* can take any value as the coefficient of *x* in the given plane is 0.	1